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An L_1-Analysis of the Iterative Proportional Fitting Procedure

Dedicato alla memoria di Bruno Simeone (1945–2010)

Friedrich Pukelsheim

Abstract Convergence of the Iterative Proportional Fitting procedure is analyzed. The input comprises a nonnegative weight matrix, and positive target marginals for rows and columns. The output sought is what is called the biproportional fit, a scaling of the input weight matrix through row and column divisors so as to equate row and column sums to target marginals. The procedure alternates between the fitting of rows, and the fitting of columns. We monitor progress with an L_1-error function measuring the distance between current row and column sums and target row and column marginals. The procedure converges to the biproportional fit if and only if the L_1-error tends zero. In case of non-convergence the procedure appears to oscillate between two accumulation points. The oscillation result is contingent on the conjecture that the even-step subsequence of the procedure is always convergent. The conjecture is established in the specific setting when the even-step subsequence has a connected accumulation point, but remains open in general.

Keywords Alternating scaling algorithm · Biproportional fitting · Entropy · Matrix scaling · RAS procedure

AMS 2010 subject classification: 62P25, 62H17

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1 Introduction

We present a novel, L_1-based analysis of the Iterative Proportional Fitting (IPF) procedure. The IPF procedure is an algorithm for scaling rows and columns of an input $k \times \ell$ weight matrix $A = ((a_{ij}))$ so that the output matrix $B = ((b_{ij}))$ achieves row sums equal to a prespecified vector of row marginals, $r = (r_1, \ldots, r_k)$, and column sums equal to a prespecified vector of column marginals, $s = (s_1, \ldots, s_\ell)$. All weights are assumed nonnegative, $a_{ij} \geq 0$, with at least one entry in each row and column of A being positive. All marginals are taken to be positive, $r_i > 0$ and $s_j > 0$.

The problem has a continuous variant, the biproportional fitting problem, and a discrete variant, the biproportional apportionment problem. In the continuous variant, the entries of the output matrix B are nonnegative real numbers, $b_{ij} \in [0, \infty)$. The output B is called a biproportional fit of the weight matrix A to the target marginals r and s. The IPF procedure iteratively calculates scaled matrices $A(t) = \big(\big(a_{ij}(t)\big)\big)$, where for odd steps $t-1$ row sums are matching, $a_{i+}(t-1) = r_i$ for all $i \leq k$, while for even steps t column sums match, $a_{+j}(t) = s_j$ for all $j \leq \ell$. If a biproportional fit B exists, the sequence of scaled matrices $A(t)$, $t \geq 1$, converges to B.

In the discrete problem variant the entries of B are restricted be nonnegative integers, $b_{ij} \in \{0, 1, 2, \ldots\}$. Then the output matrix B is called a biproportional apportionment, for the weight matrix A and the target marginals r and s. The procedure to solve the discrete problem is the Alternating Scaling (AS) algorithm. At step t it produces a matrix $A(t)$ with entries $a_{ij}(t)$ not only scaled but also rounded. Due to possible ties there are (rare) instances when a biproportional apportionment B exists while the AS algorithm stalls and fails to converge to it. An example is given by Gaffke and Pukelsheim (2008b, page 157).

Our research arose from the desire to better understand the interplay between the continuous IPF procedure, and the discrete AS algorithm. The present paper focuses on the continuous fitting problem. Yet our major tool, the L_1-error function

$$f\big(A(t)\big) = \sum_{i \leq k} \Big| a_{i+}(t) - r_i \Big| + \sum_{j \leq \ell} \Big| a_{+j}(t) - s_j \Big|.$$

is borrowed from Balinski and Demange's (1989) inquiry into the discrete apportionment problem. In the discrete case the error function is quite suggestive, simply counting along rows and columns how many units step t allocates wrongly. For the continuous problem the L_1-error is, at first glance, just one out of many ways to assess lack of fit. At second glance it is a most appropriate way, as this paper endeavors to show.

1.1 The literature on biproportional fitting

The continuous biproportional fitting problem is the senior member of the two problem families. It has created an enormous body of literature of which we review only the papers that influenced the present research. The term *IPF procedure* prevails in Statistics, see Fienberg and Meyer (2006), or Speed (2005). Some Statisticians speak of *matrix raking*, such as Fagan and Greenberg (1987). In Operations Research and Econometrics the label *RAS method* is popular, pointing to a (diagonal) matrix R of row multipliers, the weight matrix A, and a (diagonal) matrix S of column multipliers, as mentioned already by Bacharach (1965, 1970). Computer scientists prefer the term *matrix scaling*, as in Rote and Zachariasen (2007).

Deming and Stephan (1940) are first to popularize the IPF procedure though there are earlier papers using the idea, as pointed out by Fienberg and Meyer (2006). Deming and Stephan (1940, page 440) recommend terminating iterations when *the table reproduces itself*, that is, in our terminology, when the scaled matrices $A(t-1)$ and $A(t)$ get close to each other. This closeness is what is measured by the L_1-error function $f\big(A(t)\big)$, see the remarks leading to our Lemma 1. While successfully advocating the merits of the algorithm, Deming and Stephan were somewhat led astray in its analysis, as communicated by Stephan (1942).

Brown (1959) proposes a convergence proof which Ireland and Kullback (1968) criticize to lack rigor. The latter authors establishes convergence by relating the IPF procedure to the minimum entropy solution. Csiszár (1975, page 155) notes that their argument is incomplete, and that the generalization to measure-spaces by Kullback (1968) suffers from a similar flaw. Csiszár (1975) salvages the entropy approach, and Rüschendorf (1995) extend it to general measure-spaces. Rüschendorf and Thomsen (1993, 1997) rectify a technical detail that escaped Csiszár's (1975) attention.

Despite of the emphasis on entropy, the ultimate arguments of Ireland and Kullback (1968, eqs. (4·32) and (4·33)) substitute convergence of entropy by convergence in L_1, referring to a result of Kullback (1966). Also Bregman (1967) starts out with entropy, and then uses the L_1-error function. Here we dispose of the entropy detour, and use L_1 from start to finish. Ireland and Kullback (1968, page 184) prove that the entropy criterion decreases monotonically, as does the likelihood function of Bishop, Fienberg and Holland (1975, page 86), and the L_1-error function, see Bregman (1967, page 197). Some of the literature replaces entropy by other criteria, as overviewed by Kalantari, Lari, Ricca and Simeone (2008). Marshall and Olkin (1968) and Macgill (1977) minimize a quadratic objective function. The proof of our Theorem 3 is inspired by Pretzel (1980) who uses a criterion related to a geometric matrix-mean.

The question when a biproportional fit exists generated a wealth of papers, such as Brualdi, Parter and Schneider (1966), Schneider (1990), and Brown, Chase and Pittinger (1993). Many of them use network and graph theory. We refer to such arguments in the proofs of Theorem 1 and 2, see also Pukelsheim, Ricca, Simeone, Scozzari and Serafini (2012) Moreover, the formula for the L_1-error limit in Theorem 4 is related to viewing the issue as a transportation problem. Rachev and Rüschendorf (1998) present an in-depth development of measure-theoretic mass transportation problems, and we tend to believe that there are more interrelations than we have been able to identify.

Fienberg (1970) opens up a different route by embedding the IPF procedure into the geometry of the manifold of constant interaction in a $(k\ell-1)$-dimensional simplex of reference. The author works with the assumption that all input weights are positive, $a_{ij} > 0$. He points out (page 915) that the extension to problems involving zero weights is *quite complex*, which indeed is attested to by much of the literature. Ireland and Kullback's (1968, page 182) plea of assuming positive weights in order to *to simplify the argument* is a friendly understatement, unless it is meant to be the utter truth.

Yet another approach, staying as close to calculus as possible, is due to Bacharach (1965, 1970), and Sinkhorn (1964, 1966, 1967, 1972, 1974) and Sinkhorn and Knopp (1967). Much of the present paper is owed to Bacharach.

Michael Owen Leslie Bacharach (b. 1936, d. 2002) was an Oxford econometrician. In 1965 he earned a PhD degree in Mathematics from Cambridge. His thesis was published as Bacharach (1965), and became Section 4 of Bacharach (1970). Richard Dennis Sinkhorn (b. 1934, d. 1995) received his Mathematics PhD in 1962 from the University of Wisconsin–Madison, with a thesis entitled *On Two Problems Concerning Doubly Stochastic Matrices*. Throughout his career he served as a Mathematics professor with the University of Houston. Though contemporaries, neither of the two ever quoted the other.

1.2 The literature on biproportional apportionment

The discrete biproportional apportionment problem is the junior problem family, first put forward by Balinski and Demange (1989), see also Balinski and Rachev (1997) and Simeone and Pukelsheim (2006). The operation of rounding scaled quantities to integers sounds most attractive for the statistical analysis of frequency tables, as noted by Wainer (1998) and Pukelsheim (1998). It disposes of any disclaimer that the adjusted figures are *rounded off, hence when summed may occasionally disagree a unit or so*, as warned in Table I of Deming and Stephan (1940, page 433). When calculating percentages, as in Table 3.6-4 of Bishop, Fienberg and Holland (1975, page 99), the method does not stop short of 99 percent. Yet Balinski's motivation was not contingency table analysis in statistics, but proportional representation systems for parliamentary elections.

The task of allocating seats of a parliamentary body to political parties does not tolerate any disclaimer excusing residual rounding errors. Methods must account for each seat. This is achieved by biproportional methods. In 2003, the Swiss Canton of Zurich adopted a doubly proportional system, the biproportional divisor method with standard rounding, see Pukelsheim and Schuhmacher (2004), Balinski and Pukelsheim (2006), and Pukelsheim and Schuhmacher (2011). The method may be attractive also for other countries, see Pennisi (2006) for Italy, Zachariassen and Zachariasen (2006) for the Farœ Islands, Ramírez, Pukelsheim, Palomares and Martínez (2008) for Spain, and Oelbermann and Pukelsheim (2011) for the European Union.

When I had the privilege of advising the Zurich politicians on the amendment of the electoral law, I felt it inappropriate to present biproportionality as a method minimizing entropy, or being justified through differential geometry of smooth manifolds in high-dimensional simplexes. The procedure simply does what proportionality is about: Scale and round! Scaling within electoral districts (rows) achieves proportionality among the parties campaigning in that district. Scaling within parties (columns) secures district lists of any party to be handled proportionally. The final rounding step is inevitable, as deputies are counted in whole numbers and do not come in fractions.

That biproportional apportionment also won administrative support is a victory of the IPF procedure. Its discrete sibling, the AS algorithm, enables officials to easily calculate district divisors and party divisors. Once suitable divisors are publicized all voters can double-check the outcome. They only need to take the vote count of the party of their choice in their district, divide it by the respective district and party divisors, and round the result to the nearest seat number. A computer program for carrying out the apportionment is provided at `www.uni-augsburg.de/bazi`, see Pukelsheim (2004), Joas (2005), Maier (2009). The user may choose to run the AS algorithm, the Tie-and-Transfer (TT) algorithm of Balinski and Demange (1989), or hybrid combinations of the two. The performance of the algorithms is studied by Maier, Zachariassen and Zachariasen (2010)

In the electoral application the entries a_{ij} in the weight matrix A signify vote counts, and the occurrence of zero weights is inevitable. It is only normal that there exists a party j not campaigning in some district i, which than enters into final evaluations with $a_{ij} = 0$. It is no longer appropriate *to simplify the argument* by assuming all weights to be positive. Zero weights must be properly dealt with, even if the labor entailed becomes quite complex.

1.3 Section overview

A brief overview of the paper is as follows. Section 2 introduces the notion of biproportional fits. If it exists, the biproportional fit is unique (Theorem 1). A biproportional fit is called direct when in its definition the transition to limits is superfluous. Theorem 2 offers five equivalent ways to check for directness. Direct biproportional fits are called *interior solutions* by Bacharach (1970, page 45), whereas his *boundary solutions* are biproportional fits that are not direct. Though not needed in the sequel, Proposition 1 shows that two matrices are biproportionally equivalent if and only if they have identical cyclical ratios. The proof projects the problem into the linear space spanned by cycle matrices, as in Gaffke and Pukelsheim (2008a, page 178).

Section 3 describes the IPF procedure. The generated IPF sequence $A(t)$, $t \geq 1$, has a nonincreasing L_1-error that is bounded by

$$f\big(A(t)\big) \geq \max_{I \subseteq \{1,\ldots,k\}} \Big(r_I - s_{J_A(I)} + s_{J_A(I)'} - r_{I'}\Big),$$

where r_I and $s_{J_A(I)}$ are partial sums of row and column marginals, and a prime indicates set complements (Lemma 1). Theorem 3 presents necessary and sufficient conditions for convergence, $\lim_{t\to\infty} A(t) = B$. In case of convergence the limit B is the biproportional fit sought. In Section 4 three examples illustrate the results.

Section 5 scrutinizes the L_1-error limit when the IPF procedure does not converge. If the even-step subsequence converges to a limit, then so does the odd-step subsequence, and these limits are the only two accumulation points of the full IPF sequence (Lemma 2). For these cases Theorem 4 proves that the limiting L_1-error attains the lower bound displayed above. Using a set of inequalities of Bacharach (1970), Theorem 5 shows that the even-step subsequence is convergent in the specific instance when it admits an accumulation point that is connected. There is ample evidence from empirical and simulation data to conjecture that the even-step subsequence is convergent also in all other cases, but we are unable to present a general proof.

1.4 Notation

A plus-sign is used as a subscript to indicate summation over the index that otherwise appears in its place, as in $r_+ = \sum_{i\leq k} r_i$, $s_+ = \sum_{j\leq \ell} s_j$, or $a_{++} = \sum_{i\leq k}\sum_{j\leq \ell} a_{ij}$. Partial sums are written with the range of summation in place of the index, $r_I = \sum_{i\in I} r_i$, $s_J = \sum_{j\in J} s_j$, or $a_{I\times J} = \sum_{i\in I}\sum_{j\in J} a_{ij}$. A prime signifies the complement of a set, $I' = \{1,\ldots,k\} \setminus I$. The columns connected in a matrix A to a row subset I are assembled in the subset $J_A(I) = \big\{\, j \leq \ell \,\big|\, a_{ij} \neq 0 \text{ for some } i \in I \,\big\}$.

2 Biproportional fits

A $k \times \ell$ matrix $A = ((a_{ij}))$ is called a *weight matrix* when its entries are nonnegative and no row nor column vanishes. Let $r = (r_1, \ldots, r_k)$ and $s = (s_1, \ldots, s_\ell)$ be vectors with positive entries, called *target row marginals* and *target column marginals.*

Definition (a) A $k \times \ell$ matrix $B = ((b_{ij}))$ is termed a *biproportional scaling of A* when for all rows $i \le k$ and for all columns $j \le \ell$ there exist sequences of positive *row divisors* $x_i(1)$, $x_i(2)$, ... and of positive *column divisors* $y_j(1)$, $y_j(2)$, ... satisfying

$$b_{ij} = \lim_{t\to\infty} \frac{a_{ij}}{x_i(t)y_j(t)}.$$

(b) A $k \times \ell$ matrix $B = ((b_{ij}))$ is said to *match the target marginals r and s* when its rows $i \le k$ sum to $b_{i+} = r_i$ and its columns $j \le \ell$ to $b_{+j} = s_j$.

(c) A $k \times \ell$ matrix B is called a *biproportional fit of the weight matrix A to the target marginals r and s*, when B is a biproportional scaling of A matching the target marginals r and s. □

The notation $B(A, r, s)$ would exhibit the input needed for a biproportional fit more visibly, but is dismissed as too cumbersome. If there exists a matrix B matching the target marginals r and s, then marginal totals coincide, $r_+ = b_{++} = s_+$.

Theorem 1 (Uniqueness) *There exists at most one biproportional fit B of the weight matrix A to the target marginals r and s.*

Proof Assuming two distinct biproportional fits, $B \ne C$, their difference $B - C$ is nonzero and has vanishing row and column sums. We construct a cycle of cells

$$(i_1, j_1),\ (i_1, j_2),\ (i_2, j_2),\ (i_2, j_3),\ \ldots,\ (i_{q-1}, j_{q-1}),\ (i_{q-1}, j_q),\ (i_q, j_q),\ (i_q, j_1) \quad \text{[CC]}$$

along which the entries in $B - C$ are alternatingly positive or negative. First we assemble a "long list" of cells (i_1, j_1), (i_2, j_2), ..., (i_Q, j_Q), as follows. We start with a cell (i_1, j_1) where $b_{i_1j_1} > c_{i_1j_1}$. In row i_1 there is a cell (i_1, j_2) with $b_{i_1j_2} < c_{i_1j_2}$. Next we search in column j_2 a row i_2 where $b_{i_2j_2} > c_{i_2j_2}$. Then we look for a column j_3 such that $b_{i_2j_3} > c_{i_2j_3}$. The long list terminates when encountering a column j_Q already listed, that is, when for some $P < Q$ we find $j_Q = j_P$. The initial $P - 1$ cells are discarded, and the remaining "short list" is relabeled as in [CC].

A *cyclic ratio* in a matrix is a ratio having the entries along a given cell cycle alternatingly appear in the denominator and in the numerator. Since $a_{ij} = 0$ implies $b_{ij} = c_{ij} = 0$, the cell cycle [CC] touches only upon positive entries of the weight matrix A. Let $x_i(t)$ and $y_j(t)$ denote the divisor sequences for B, and $u_i(t)$ and $v_j(t)$ for C. As biproportionality preserves cyclic ratios, the cyclic ratios in A, B, and C are seen to be equal,

$$\prod_{p\le q} \frac{a_{i_pj_{p+1}}}{a_{i_pj_p}} = \prod_{p\le q} \frac{\frac{a_{i_pj_{p+1}}}{x_{i_p}(t)y_{j_{p+1}}(t)}}{\frac{a_{i_pj_p}}{x_{i_p}(t)y_{j_p}(t)}} = \prod_{p\le q} \frac{b_{i_pj_{p+1}}}{b_{i_pj_p}} = \prod_{p\le q} \frac{\frac{a_{i_pj_{p+1}}}{u_{i_p}(t)v_{j_{p+1}}(t)}}{\frac{a_{i_pj_p}}{u_{i_p}(t)v_{j_p}(t)}} = \prod_{p\le q} \frac{c_{i_pj_{p+1}}}{c_{i_pj_p}},$$

where $j_{q+1} = j_1$. The first and third equation signs are obvious. The second equality involves a passage to the limit as t tends to ∞ and is justified since, by construction, the limiting denominator is positive, $b_{i_p j_p} > c_{i_p j_p} \geq 0$. As the left hand side is positive, the numerator must be positive, too, $b_{i_p j_{p+1}} > 0$. A similar argument establishes the last equality, with the roles of numerator and denominator interchanged.

However, the construction of the cycle [CC] precludes equality,

$$\prod_{p \leq q} \frac{b_{i_p j_{p+1}}}{b_{i_p j_p}} < \prod_{p \leq q} \frac{c_{i_p j_{p+1}}}{c_{i_p j_p}}.$$

Hence the assumption $B \neq C$ is untenable and uniqueness obtains, $B = C$. $\square$

The particular case when the transition to limits can be disposed of warrants some pertinent terminology. A biproportional scaling B is termed *direct* when its associated divisor sequences can be chosen to be constant, that is, for all rows $i \leq k$ and for all columns $j \leq \ell$ there are positive divisors u_i and v_j such that $b_{ij} = a_{ij}/(u_i v_j)$.

Directness transpires to be closely related to the notion of connectedness. A nonzero matrix C is said to be *connected* when it is not disconnected. A nonzero matrix D is called *disconnected* when a suitable permutation of rows and a suitable permutation of columns give rise to a row subset I and a column subset J such that D acquires block format,

$$D = \begin{array}{c} \\ I \\ I' \end{array} \begin{array}{c} \begin{array}{cc} J & J' \end{array} \\ \begin{pmatrix} D^{(1)} & 0 \\ 0 & D^{(2)} \end{pmatrix} \end{array},$$

where at least one of the subsets I or J is nonempty and proper, $\emptyset \subsetneq I \subsetneq \{1, \ldots, k\}$ or $\emptyset \subsetneq J \subsetneq \{1, \ldots, \ell\}$. In most applications both subsets are nonempty and proper.

We say that a matrix B *preserves the zeros of* A when all zeros of A are zeros also of B, $a_{ij} = 0 \Rightarrow b_{ij} = 0$. Two matrices A and B *have the same zeros* when $a_{ij} = 0 \Leftrightarrow b_{ij} = 0$. For keeping track of the nonzero entries in the weight matrix A we associate with every row subset $I \subseteq \{1, \ldots, k\}$ the *set of columns connected in* A *to* I,

$$J_A(I) = \left\{ j \leq \ell \,\middle|\, a_{ij} \neq 0 \text{ for some } i \in I \right\}.$$

The complement $J_A(I)'$ embraces the columns j with entries $a_{ij} = 0$ for all $i \in I$. Hence the $I \times J_A(I)'$ submatrix of A vanishes and the sum of its entries is zero, $a_{I \times J_A(I)'} = 0$. The extreme settings provide simple examples. If we choose $I = \emptyset$ then clearly $J_A(I) = \emptyset$. If $I = \{1, \ldots, k\}$ then we get $J_A(I) = \{1, \ldots, \ell\}$, since no row nor column of A vanishes.

Given a biproportional fit B, there are various ways to check for directness.

Theorem 2 (Directness) *Assume that the weight matrix A is connected, and that B is its biproportional fit to the target marginals r and s.*

Then the following five statements are equivalent:

(1) *The biproportional fit B is direct.*

(2) *The matrices A and B have the same zeros.*

(3) *There exists a nonnegative $k \times \ell$ matrix C matching the target marginals r and s such that A and C have the same zeros.*

(4) *Marginal partial sums fulfill $r_I < s_{J_A(I)}$ for all row subsets I that are nonempty and proper, $\emptyset \subsetneq I \subsetneq \{1, \ldots, k\}$.*

(5) *The biproportional fit B is connected.*

Proof (1) $\Rightarrow$ (2). A direct fit, $b_{ij} = a_{ij}/(u_i v_j)$, visibly has the same zeros as A.

(2) $\Rightarrow$ (3). The fit B, sharing all zeros with A, qualifies for a matrix C as in (3).

(3) $\Rightarrow$ (4). For every row subset I we have $a_{I \times J_A(I)'} = 0$, and hence $c_{I \times J_A(I)'} = 0$. If I is nonempty and proper then $c_{I' \times J_A(I)} > 0$, as otherwise C is disconnected and so would be A. We get $r_I = c_{I \times J_A(I)} < c_{I \times J_A(I)} + c_{I' \times J_A(I)} = s_{J_A(I)}$.

(4) $\Rightarrow$ (5). The proof is by contraposition. Assuming the biproportional fit B to be disconnected, we show that (4) is negated (I), or that biproportionality cannot hold true (II). To this end we partition B into its $K \geq 2$ connected components $I_m \times J_m$,

$$B = \begin{array}{c} \\ I_1 \\ I_2 \\ \vdots \\ I_K \end{array} \begin{array}{c} \begin{array}{cccc} J_1 & J_2 & \cdots & J_K \end{array} \\ \begin{pmatrix} B^{(1)} & 0 & \cdots & 0 \\ 0 & B^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B^{(K)} \end{pmatrix} \end{array},$$

and impose this partitioning upon the weight matrix A. The subsets $I_1, \ldots, I_K$ are nonempty and, since $K \geq 2$, proper.

I. Consider the case $a_{I_m \times J_p} = 0$, for some $m \in \{1, \ldots, K\}$ and for all $p \neq m$. With $a_{I_m \times J_m'} = \sum_{p \neq m} a_{I_m \times J_p} = 0$, the set J_m comprises the columns connected in A to I_m, that is, $J_m = J_A(I_m)$. The identity $r_{I_m} = s_{J_m} = s_{J_A(I_m)}$ negates (4).

II. Otherwise every $m \in \{1, \ldots, K\}$ comes with a *successor* $p(m) \neq m$ such that $a_{I_m \times J_{p(m)}} > 0$. But $b_{I_m \times J_{p(m)}} = 0$, whence such a block $I_m \times J_{p(m)}$ is said to be *fading*.

First we assemble a "long list" of block indices $(m_1, \ldots, m_Q)$. Beginning the list with $m_1 = 1$, we append the successor $m_{i+1} = p(m_i)$ as long as m_{i+1} is a novel contribution to the list already assembled. We terminate with m_Q as soon as its successor is already listed, $p(m_Q) \in \{m_1, \ldots, m_Q\}$. The section $(m_{p(m_Q)}, \ldots, m_Q)$ constitutes the "short list" of interest, and is relabeled $(m_1, \ldots, m_q)$. It induces in B a cycle of blocks alternating between on-diagonal blocks $I_{m_p} \times J_{m_p}$ that are connected, and off-diagonal blocks $I_{m_p} \times J_{m_{p+1}}$ that are fading, for $p \leq q$ (where $m_{q+1} = m_1$).

Second we choose a cycle of cells that is nested in the block cycle constructed. Entering some on-diagonal block $B^{(m_p)}$ in row i_p we wish to leave it through column j_p. Since the component $B^{(m_p)}$ is connected, we may have to visit perhaps not just one positive entry in $B^{(m_p)}$ before exiting column j_p, but three, or five, or another odd number $n(p)$. For notational simplicity we take $n(p) = 1$, meaning that we right away hit a positive entry, $b_{i_p,j_p} > 0$. Thus the cell cycle takes the form (i_1, j_1), ..., (i_q, j_q), with $i_p \in I_{m_p}$ and $j_p \in J_{m_p}$ for all $p \leq q$, as follows. In every off-diagonal block we select a cell $(i_p, j_{p+1}) \in I_{m_p} \times J_{m_{p+1}}$ (where $j_{q+1} = j_1$) such that $a_{i_p,j_{p+1}} > 0 = b_{i_p,j_{p+1}}$, and in every on-diagonal block we touch upon a cell with $b_{i_p,j_p} > 0$. As biproportionality preserves cyclic ratios, the cyclic ratios in A and B are equal,

$$\prod_{p \leq q} \frac{a_{i_p j_{p+1}}}{a_{i_p j_p}} = \prod_{p \leq q} \frac{\frac{a_{i_p j_{p+1}}}{x_{i_p}(t) y_{j_{p+1}}(t)}}{\frac{a_{i_p j_p}}{x_{i_p}(t) y_{j_p}(t)}} = \prod_{p \leq q} \frac{b_{i_p j_{p+1}}}{b_{i_p j_p}}.$$

However, the left hand side is positive and the right hand side is zero. Hence equality is impossible, and case II cannot materialize.

(5) $\Rightarrow$ (1). The divisor sequences that come with B may be standardized so that the first row acquires the constant divisor unity, $\widetilde{x}_1(t) = 1$, in that $\widetilde{x}_i(t) = x_i(t)/x_1(t)$ and $\widetilde{y}_j(t) = x_1(t) y_j(t)$ satisfy

$$b_{ij} = \lim_{t \to \infty} \frac{a_{ij}}{x_i(t) y_j(t)} = \lim_{t \to \infty} \frac{a_{ij}}{\widetilde{x}_i(t) \widetilde{y}_j(t)}.$$

The desired row divisors u_i and column divisors v_j are constructed via a scanning process, repeatedly using that positive entries in B necessitate positive entries in A.

In step 1 we scan the first row and equip it with divisor unity, $u_1 = 1 = \lim_{t\to\infty} \widetilde{x}_1(t)$. In step 2 we scan all columns j where $b_{1j} > 0$, and define

$$0 < v_j = \frac{a_{1j}}{u_1 b_{1j}} = \frac{\lim_{t\to\infty} \widetilde{x}_1(t) \widetilde{y}_j(t)}{\lim_{t\to\infty} \widetilde{x}_1(t)} = \lim_{t \to \infty} \widetilde{y}_j(t), \qquad \text{whence } b_{1j} = \frac{a_{1j}}{u_1 v_j}.$$

Step 3 scans all unscanned rows i with $b_{ij} > 0$ for some scanned column j, and defines

$$0 < u_i = \frac{a_{ij}}{b_{ij} v_j} = \frac{\lim_{t\to\infty} \widetilde{x}_i(t) \widetilde{y}_j(t)}{\lim_{t\to\infty} \widetilde{y}_j(t)} = \lim_{t \to \infty} \widetilde{x}_i(t), \qquad \text{whence } b_{ij} = \frac{a_{ij}}{u_i v_j}.$$

The process enlarges the scanned sets of rows and columns for at most $k + \ell$ steps.

The terminal scanned row set I and column set J put B into block format,

$$B = \begin{array}{c} \\ I \\ I' \end{array} \!\!\begin{array}{c} \begin{array}{cc} J & J' \end{array} \\ \begin{pmatrix} B^{(1)} & 0 \\ 0 & B^{(2)} \end{pmatrix} \end{array}.$$

Connectedness of B lets the scanned sets be exhaustive, $I = \{1, \ldots, k\}$ and $J = \{1, \ldots, \ell\}$. All rows and all columns having constant divisors, the fit is direct. □

Suppose that the biproportional fit B has K connected components given by the $I_m \times J_m$ blocks $B^{(m)}$, $m \leq K$, as displayed in the preceding proof. Imposing the decomposition of B upon A, let $A^{(m)}$ be the $I_m \times J_m$ block extracted from A. The divisors of A serve as divisors for $A^{(m)}$ as well, $\lim_{t\to\infty} a_{ij}^{(m)} / \big(x_i(t) y_j(t)\big) = b_{ij}^{(m)}$ for all $i \in I_m$ and $j \in J_m$. Thus the fitting of the partial weight matrix $A^{(m)}$ to the partial marginals $(r_i)_{i \in I_m}$ and $(s_j)_{j \in J_m}$ yields the direct biproportional fit $B^{(m)}$, by Theorem 2(5). Defining the *trimmed weight matrix* A^B to consists of the blocks $A^{(1)}, \ldots, A^{(K)}$ and zeros elsewhere, the fitting of A^B to the target marginals r and s yields the biproportional fit B which is direct.

We adjoin a proposition, not needed in the sequel, to elucidate the interplay of biproportional scalings and cyclic ratios. Two weight matrices A and B are said to be *biproportionally equivalent* when they are direct biproportional scalings of each other, that is, for all rows $i \le k$ and for all columns $j \le \ell$ there are positive divisors u_i and v_j fulfilling $b_{ij} = a_{ij}/(u_i v_j)$. The *support set* of a matrix A is constituted by the cells where A has nonzero entries, $\mathrm{supp}(A) = \big\{(i,j) \in \{1,\dots,k\} \times \{1,\dots,\ell\} \,\big|\, a_{ij} \neq 0\big\}$. If A and B are biproportionally equivalent, then they have the same support sets.

Let $S \subseteq \{1,\dots,k\} \times \{1,\dots,\ell\}$ denote a subset of cells. A *cell cycle on S* is defined to consist of a sequence of $2q$ cells, as in display [CC] in the proof of Theorem 1, involving some $q \ge 2$ distinct rows $i_1,\dots,i_q$ and distinct columns $j_1,\dots,j_q$ satisfying $(i_p, j_p) \in S$ and $(i_p, j_{p+1}) \in S$ for all $p \le q$. We adopt the convention that always $j_{q+1} = j_1$. Two weight matrices A and B are said to be *cyclically equivalent* when they share a common support set, $\mathrm{supp}(A) = \mathrm{supp}(B) = S$ say, and all cell cycles on S fulfill $\prod_{p \le q} a_{i_p j_{p+1}}/a_{i_p j_p} = \prod_{p \le q} b_{i_p j_{p+1}}/b_{i_p j_p}$.

Proposition 1 (Equivalence) *Let A and B be any two weight matrices.*

Then A and B are biproportionally equivalent if and only if A and B are cyclically equivalent.

Proof As in the proofs of Theorems 1 and 2, the direct part is a one-liner,

$$\prod_{p \le q} \frac{a_{i_p j_{p+1}}}{a_{i_p j_p}} = \prod_{p \le q} \frac{\frac{a_{i_p j_{p+1}}}{u_{i_p} v_{j_{p+1}}}}{\frac{a_{i_p j_p}}{u_{i_p} v_{j_p}}} = \prod_{p \le q} \frac{b_{i_p j_{p+1}}}{b_{i_p j_p}}.$$

For the converse part let S denote the support set common to A and B. We need to establish the existence of some positive numbers u_i and v_j such that $a_{ij}/b_{ij} = u_i v_j$. That is, we are looking for solutions $x_i = \log u_i$ and $y_j = \log v_j$ to the system of linear equations $\log(a_{ij}/b_{ij}) = x_i + y_j$, $(i,j) \in S$.

Denoting by E_{ij} the $k \times \ell$ Euclidean unit matrix with entry unity in cell (i,j) and zeros elsewhere, we work in the linear space $V = \mathrm{span}\big\{E_{ij} \,\big|\, (i,j) \in S\big\}$ with inner product $\langle C, D\rangle = \mathrm{trace}\, C'D$. Consider the subspace

$$L = \Big\{\sum\nolimits_{(i,j)\in S} (x_i + y_j) E_{ij} \,\Big|\, x_1,\dots,x_k, y_1,\dots,y_\ell \in \mathbb{R}\Big\}.$$

We need to show that $C = \sum_{(i,j)\in S} \log(a_{ij}/b_{ij}) E_{ij}$ lies in L. Equivalently, we verify that C is orthogonal to $L^\perp$. The orthogonal complement $L^\perp$ consists of all matrices D in V having vanishing row and column sums. Indeed, the inner products

$$\Big\langle \sum_{(i,j)\in S} (x_i + y_j) E_{ij}, D \Big\rangle = \sum_{(i,j)\in S} (x_i + y_j) d_{ij} = \sum_{i \le k} x_i d_{i+} + \sum_{j \le \ell} y_j d_{+j}$$

vanish for all x_i and y_j if and only if all $d_{i+} = 0$ and $d_{+j} = 0$.

For a cell cycle $(i_1, j_1), \dots, (i_q, j_q)$ the *cycle matrix* $D\big((i_1,j_1),\dots,(i_q,j_q)\big) = \sum_{p \le q} \big(E_{i_p j_p} - E_{i_p j_{p+1}}\big)$ is defined to have entry 1 in cells (i_p, j_p) and entry -1 in cells (i_p, j_{p+1}). The cycle matrices from cell cycles in S provide a spanning set for $L^\perp$,

$$L^\perp = \mathrm{span}\Big\{D\big((i_1,j_1),\dots,(i_q,j_q)\big) \,\Big|\, (i_1,j_1),\dots,(i_q,j_q) \text{ cell cycle in } S\Big\}.$$

Evidently the right hand subspace is included in $L^{\perp}$ since every cycle matrix has all row and column sums equal to zero. Conversely, every nonzero matrix $B \in L^{\perp}$ can be represented as a linear combination of cycle matrices, as follows. Since B has vanishing row and column sums, we may identify a first cell cycle in $\operatorname{supp}(B) \subseteq S$ by proceeding just as in the proof of Theorem 1. The first cell cycle induces a cycle matrix $D(1)$ with $\operatorname{supp}\big(D(1)\big) \subseteq \operatorname{supp}(B)$. We choose some cell (i,j) in the support of $D(1)$ and set $\lambda(1) = b_{ij}/d_{ij}(1) \neq 0$. Now $B(1) = B - \lambda(1)D(1)$ has a support set strictly smaller than that of B, $\operatorname{supp}\big(B(1)\big) \subsetneq \operatorname{supp}(B)$. In case $B(1)$ is a cycle matrix or zero, $B = \lambda(1)D(1) + B(1)$ lies in the right hand span. Otherwise, the reduction processes is applied to $B(1)$, and continues with $B = \lambda(1)D(1) + \lambda(2)D(2) + B(2)$. The reduction process may have to be repeated, but terminates after finitely many steps.

Therefore it suffices to show that $C = \sum_{(i,j)\in S} \log(a_{ij}/b_{ij})E_{ij}$ is orthogonal to every cycle matrix $D = \sum_{p\le q}\big(E_{i_pj_p} - E_{i_pj_{p+1}}\big)$. But this follows from cyclic equivalence,

$$\langle C, D\rangle = \sum_{p\le q}\Big(c_{i_pj_p} - c_{i_pj_{p+1}}\Big) = \log\prod_{p\le q}\frac{a_{i_pj_p}}{b_{i_pj_p}}\frac{b_{i_pj_{p+1}}}{a_{i_pj_{p+1}}} = \log 1 = 0. \qquad \square$$

3 The IPF procedure

The IPF procedure is an algorithm for determining an existing biproportional fit of the weight matrix A to the target marginals r and s. It operates alternatingly on the rows and on the columns of A, in that odd steps scale rows to match target row marginals, while even steps scale columns to match target column marginals.

If the biproportional fit exists, marginal totals coincide, $r_+ = s_+$. It would seem just natural to assume equality of marginal totals from the very beginning. We do not do so, since the IPF procedure may well be run with target marginals not sharing the same total, and since the procedure cannot evade different marginal (sub-)totals of connected components when decomposing limit matrices of convergent subsequences.

The IPF procedure is initialized by scaling the given weight matrix A into a matrix $A(0)$ which has column sums equal to target column marginals. The initialization routine uses column divisors $\sigma_j(0) = a_{+j}/s_j$ and sets $a_{ij}(0) = a_{ij}/\sigma_j(0)$, for all columns $j \le \ell$ and rows $i \le k$. This fits columns, $a_{+j}(0) = s_j$, and the sum of the initialized weights becomes $a_{++}(0) = s_+$. Thereafter the procedure advances in pairs of an odd step $t-1$ and an even step t, for $t = 2, 4, \ldots$:

- Odd steps $t-1$ fit row sums to target row marginals, by calculating row divisors $\rho_i(t-1)$ from the preceding even step $t-2$ and then defining scaled weights $a_{ij}(t-1)$:

$$\rho_i(t-1) = \frac{a_{i+}(t-2)}{r_i}, \qquad \text{[IPF1]}$$

$$a_{ij}(t-1) = \frac{a_{ij}(t-2)}{\rho_i(t-1)}, \qquad \text{[IPF2]}$$

for all rows $i \le k$ and for all columns $j \le \ell$.

- Even steps t fit column sums to target column marginals, by calculating column divisors $\sigma_j(t)$ from the preceding odd step $t-1$ and then defining scaled weights $a_{ij}(t)$:

$$\sigma_j(t) = \frac{a_{+j}(t-1)}{s_j}, \qquad \text{[IPF3]}$$

$$a_{ij}(t) = \frac{a_{ij}(t-1)}{\sigma_j(t)}, \qquad \text{[IPF4]}$$

for all columns $j \le \ell$ and for all rows $i \le k$.

All divisors stay positive since no row nor column of A is allowed to vanish. Definitions [IPF1] and [IPF3] are reminiscent of likelihood ratios, of fitted distributions relative to target distributions. When weighted by their corresponding marginal distributions, row and column divisors have means that are ratios of marginal totals,

$$\sum_{i\le k} \rho_i(t-1)\,\frac{r_i}{r_+} = \frac{a_{++}(t-2)}{r_+} = \frac{s_+}{r_+}, \qquad \sum_{j\le \ell} \sigma_j(t)\,\frac{s_j}{s_+} = \frac{a_{++}(t-1)}{s_+} = \frac{r_+}{s_+}.$$

The products of consecutive divisors have mean unity when averaged relative to the product of the marginal distributions,

$$\sum_{i\le k}\sum_{j\le \ell} \rho_i(t-1)\sigma_j(t)\,\frac{r_i}{r_+}\frac{s_j}{s_+} = \frac{s_+}{r_+}\frac{r_+}{s_+} = 1.$$

The mean is unity also when the average is taken relative to the probability distribution $(1/s_+)A(t)$. Indeed, with $a_{ij}(t-2) = \rho_i(t-1)\sigma_j(t)a_{ij}(t)$ from [IPF2] and [IPF4] we get

$$\frac{1}{s_+}\sum_{i\le k}\sum_{j\le \ell} \rho_i(t-1)\sigma_j(t)\,a_{ij}(t) = \frac{a_{++}(t-2)}{s_+} = 1.$$

In steps [IPF1] and [IPF3], the IPF procedure generates *incremental row divisors* $\rho_i(1), \rho_i(3), \ldots$, and *incremental column divisors* $\sigma_j(2), \sigma_j(4), \ldots$. They give rise to the *cumulative divisors* $x_i(t)$ and $y_j(t)$, defined for $t = 2, 4, \ldots$ through

$$\rho_i(1)\,\rho_i(3)\,\cdots\,\rho_i(t-1) = x_i(t-1) = x_i(t),$$
$$\sigma_j(0)\,\sigma_j(2)\,\sigma_j(4)\,\cdots\,\sigma_j(t) = y_j(t) = y_j(t+1).$$

Adjoining $y_j(1) = \sigma_j(0)$, cumulative divisors $x_i(t)$ and $y_j(t)$ are defined for all $t \ge 1$.

The scaled weights thus take the form $a_{ij}(t) = a_{ij}/\big(x_i(t)y_j(t)\big)$. Therefore every *scaled weight matrix* $A(t) = \big(\big(a_{ij}(t)\big)\big)$ is seen to be a (direct) biproportional scaling of A. In odd steps $t-1$, the row sums of $A(t-1)$ match the target row marginals. In even steps t, the column sums of $A(t)$ match the target column marginals.

The matrix sequence $A(t)$, $t \ge 1$, is called the *IPF sequence, for the fitting of the weight matrix A to the target marginals r and s*. Interest focuses on the instances when the IPF procedure is convergent, $\lim_{t\to\infty} A(t) = B$ say. Then the limit B is the unique biproportional fit, by Theorem 1. Directness may be checked using Theorem 2.

For monitoring progress of the IPF procedure we use the L_1*-error function* f that measures in rows and columns the deviation of current sums from target marginals,

$$f\big(A(t)\big) = \sum_{i\le k}\Big|a_{i+}(t) - r_i\Big| + \sum_{j\le \ell}\Big|a_{+j}(t) - s_j\Big|.$$

Odd steps $t-1$ have row sums matching their target marginals, whence the L_1-error $f\big(A(t-1)\big)$ is equal to the (second) column-error sum. For even steps t, the (first) row-error sum is decisive.

The L_1-error function admits another interpretation, as the L_1-distance between a scaled weight matrix and its successor. To see this for an even step t, we substitute $r_i = a_{i+}(t)/\rho_i(t+1)$ from [IPF1] and $a_{ij}(t)/\rho_i(t+1) = a_{ij}(t+1)$ from [IPF2] to obtain

$$f\big(A(t)\big) = \sum_{i\le k}\left|1 - \frac{1}{\rho_i(t+1)}\right| a_{i+}(t) = \sum_{i\le k}\sum_{j\le \ell}\Big|a_{ij}(t) - a_{ij}(t+1)\Big|.$$

Definitions [IPF3] and [IPF4] yield the result for odd steps $t-1$. Deming and Stephan (1940, page 440) recommend that the IPF procedure *is continued until the table reproduces itself.* This is exactly what is captured by the error function f: The table reproduces itself, $A(t) = A(t+1)$, if and only if the L_1-error is zero, $f\big(A(t)\big) = 0$. The L_1-error function behaves quite reasonable also in other respects.

Lemma 1 (Monotonicity) *The L_1-error function is nonincreasing, $f\big(A(t-1)\big) \ge f\big(A(t)\big)$ for every step $t \ge 1$, and bounded from below according to*

$$f\big(A(t)\big) \ge \max_{I\subseteq\{1,\ldots,k\}}\Big(r_I - s_{J_A(I)} + s_{J_A(I)'} - r_{I'}\Big).$$

Proof Let step $t \ge 2$ be even. Substituting $r_i = a_{i+}(t-1) = \sum_{j\le\ell}\sigma_j(t)a_{ij}(t)$ from [IPF4] and $\sigma_j(t)s_j = a_{+j}(t-1)$ from [IPF3], the triangle inequality yields

$$f\big(A(t)\big) = \sum_{i\le k}\left|\sum_{j\le \ell}\Big(1-\sigma_j(t)\Big)a_{ij}(t)\right| \le \sum_{j\le\ell}\Big|\sigma_j(t) - 1\Big| s_j = f\big(A(t-1)\big).$$

With definitions [IPF1] and [IPF2] the argument carries over to odd steps $t \ge 1$. Thus monotonicity is established.

The bound is derived as follows. Assuming step $t \ge 2$ to be even, columns are fitted. Let $I \subseteq \{1,\ldots,k\}$ be an arbitrary subset of rows. Since $\sum_{i\in I}\big(a_{i+}(t) - r_i\big) + \sum_{i\in I'}\big(a_{i+}(t) - r_i\big) = \sum_{i\le k}\big(a_{i+}(t) - r_i\big) = s_+ - r_+$, the complement I' satisfies

$$\sum_{i\in I'}\Big(a_{i+}(t) - r_i\Big) = s_+ - r_+ + \sum_{i\in I}\Big(r_i - a_{i+}(t)\Big).$$

Neglecting absolute values we get

$$f\big(A(t)\big) \ge \sum_{i\in I}\Big(r_i - a_{i+}(t)\Big) + \sum_{i\in I'}\Big(a_{i+}(t) - r_i\Big) = s_+ - r_+ + 2\sum_{i\in I}\Big(r_i - a_{i+}(t)\Big).$$

The sum is decomposed according to

$$\begin{aligned}\sum_{i\in I}\Big(r_i - a_{i+}(t)\Big) &= r_I - \left(\sum_{i\in I}\sum_{j\in J_A(I)} + \sum_{i\in I}\sum_{j\in J_A(I)'} + \sum_{i\in I'}\sum_{j\in J_A(I)} - \sum_{i\in I'}\sum_{j\in J_A(I)}\right) a_{ij}(t) \\ &= r_I - s_{J_A(I)} - a_{I\times J_A(I)'}(t) + a_{I'\times J_A(I)}(t) \\ &\geq r_I - s_{J_A(I)} - 0 + 0.\end{aligned}$$

In the last line, the identity $a_{I\times J_A(I)'}(t) = 0$ is inherited from the input, $a_{I\times J_A(I)'} = 0$. The estimate $a_{I'\times J_A(I)}(t) \geq 0$ holds because all weights are nonnegative. This yields the bound $f\big(A(t)\big) \geq s_+ - r_+ + 2\left(r_I - s_{J_A(I)}\right) = r_I - s_{J_A(I)} + s_{J_A(I)'} - r_{I'}$, for every row subset I. The transition to the maximum tightens these bounds as much as possible. □

From $I = \{1, \ldots, k\}$ and $I = \emptyset$ we obtain the looser bound $|r_+ - s_+|$. In case of convergence the limiting error vanishes, of course, and so do any lower bounds.

Theorem 3 (Convergence) *The following five statements are equivalent:*

(1) *The IPF sequence $A(t)$, $t \geq 1$, is convergent.*

(2) *The biproportional fit of the weight matrix A to the targets marginals r and s exists.*

(3) *There exists a nonnegative $k \times \ell$ matrix C matching the target marginals r and s such that C preserves the zeros of A.*

(4) *Marginal totals are equal, $r_+ = s_+$, and marginal partial sums fulfill $r_I \leq s_{J_A(I)}$ for all row subsets $I \subseteq \{1, \ldots, k\}$.*

(5) *The IPF sequence $A(t)$, $t \geq 1$, has L_1-errors tending to zero, $\lim_{t\to\infty} f\big(A(t)\big) = 0$.*

Proof (1) ⇒ (2). If the IPF sequence converges then its limit B is a biproportional scaling of A. It inherits matching row sums along odd steps and matching column sums along even steps. By Theorem 1, B is the biproportional fit.

(2) ⇒ (3). The biproportional fit clearly qualifies for a matrix C required in (3).

(3) ⇔ (4). This equivalence is the Gale/Hoffman Feasible Distribution Theorem of network theory, see Rockafellar (1984, page 69) or Berge (1985, page 82).

(3) ⇒ (5). Our argument is inspired by Pretzel (1980). With a matrix C as in (3), let $g\big(A(t)\big)$ be the geometric matrix-mean of the entries $a_{ij}(t)$, with exponents c_{ij}/c_{++},

$$g\big(A(t)\big) = \prod_{i\leq k}\prod_{j\leq\ell} a_{ij}(t)^{\frac{c_{ij}}{c_{++}}}.$$

A base zero comes with exponent zero, $a_{ij}(t) = 0 \Rightarrow a_{ij} = 0 \Rightarrow c_{ij} = 0$, and contributes the factor $0^0 = 1$. Therefore all means stay positive, $g\big(A(t)\big) > 0$.

On the way-up from an even step $t-2$ via $t-1$ to the subsequent even step t, definitions [IPF2] and [IPF4] yield $a_{ij}(t-2) = \rho_i(t-1)\sigma_j(t)a_{ij}(t)$. With $c_{i+} = r_i$, $c_{+j} = s_j$, and $c_{++} = r_+ = s_+$, we obtain

$$g\big(A(t-2)\big) = \left(\prod_{i\leq k}\rho_i(t-1)^{\frac{r_i}{r_+}}\right)\left(\prod_{j\leq\ell}\sigma_j(t)^{\frac{s_j}{s_+}}\right) g\big(A(t)\big) \leq g\big(A(t)\big).$$

The estimate follows from the geometric-arithmetic-mean inequalities of the divisors,

$$\prod_{i\leq k}\rho_i(t-1)^{\frac{r_i}{r_+}} \leq \sum_{i\leq k}\rho_i(t-1)\frac{r_i}{r_+} = \frac{s_+}{r_+} = 1, \qquad \prod_{j\leq \ell}\sigma_j(t)^{\frac{s_j}{s_+}} \leq \sum_{j\leq \ell}\sigma_j(t)\frac{s_j}{s_+} = \frac{r_+}{s_+} = 1.$$

Therefore the even-step matrix-mean subsequence is positive, isotonic, and bounded, $0 < g\big(A(t-2)\big) \leq g\big(A(t)\big) \leq c_{++}$, and hence converges to a nonzero and finite value. It follows that, in the limit, the geometric-arithmetic-mean inequalities hold with equality, whence the averaged quantities become equal to a common value. Since their mean is unity, the common value must be unity, too. For all rows $i \leq k$ we obtain $\lim_{t=1,3,\ldots}\rho_i(t) = 1$, whence definition [IPF1] yields $\lim_{t\to\infty} a_{i+}(t) = r_i$. Similarly $\lim_{t=0,2,\ldots}\sigma_j(t) = 1$, and [IPF3] gives $\lim_{t\to\infty} a_{+j}(t) = s_j$ for all columns $j \leq \ell$. With row and column sums of $A(t)$ tending to target marginals, the limiting L_1-error is zero.

(5) $\Rightarrow$ (1). With $h = \max\{r_+, s_+\}$, the IPF sequence $A(t)$ stays in the compact set $[0,h]^{k\times\ell}$ of the matrix space $\mathbb{R}^{k\times\ell}$. Let B be an accumulation point along a subsequence $A(t_n)$, $n \geq 1$. From (5) we get $f(B) = \lim_{n\to\infty} f\big(A(t_n)\big) = \lim_{t\to\infty} f\big(A(t)\big) = 0$. With row and column sums fitted, B is a biproportional fit. By Theorem 1 there is but one. Hence the IPF sequence $A(t)$ has B for its unique accumulation point, and converges. □

Alternate proof A promising way to proceed from (3) to (5) might be as follows.

(3) $\Rightarrow$ (4). As in the proof of Theorem 2, the definition of $J_A(I)$ entails $c_{I\times J_A(I)'} = 0 \leq c_{I'\times J_A(I)}$, and $r_I = c_{I\times J_A(I)} + c_{I\times J_A(I)'} \leq c_{I\times J_A(I)} + c_{I'\times J_A(I)} = s_{J_A(I)}$.

(4) $\Rightarrow$ (5). Equal marginal totals entail $r_I - s_{J_A(I)} + s_{J_A(I)'} - r_{I'} = 2\left(r_I - s_{J_A(I)}\right)$. Once the L_1-error bound of Lemma 1 were shown to be sharp in the limit,

$$\lim_{t\to\infty} f\big(A(t)\big) = \max_{I\subseteq\{1,\ldots,k\}} 2\Big(r_I - s_{J_A(I)}\Big),$$

the step from (4) to (5) would be evident. □

Specifically, if all entries in the weight matrix A are positive and the target marginals share the same total, $r_+ = s_+ = h$ say, the matrix C with entries $c_{ij} = r_i s_j / h > 0$ satisfies statement (3) of Theorems 3 and 2. Hence the IPF sequence converges to the biproportional fit and the fit is direct, in case A is positive and $r_+ = s_+$.

4 Three examples

Previous and subsequent results are illustrated by three examples. In Example 1 the L_1-error function vanishes exponentially fast, in Example 2 the speed is linear. In Example 3 the L_1-error function has limit 2, and the IPF sequence fails to converge. The 2×2 weight matrix A and the target column marginals $s = (3,3)$ stay the same throughout, and hence so does the initialized matrix $A(0)$:

$$A = \begin{pmatrix} 30 & 0 \\ 10 & 20 \end{pmatrix}, \quad A(0) = \begin{pmatrix} \frac{9}{4} & 0 \\ \frac{3}{4} & 3 \end{pmatrix}.$$

Target row marginals r vary, all having component sum $r_+ = 6$. The examples are extended to instances where the marginals r are scaled into $\widetilde{r} = \delta r$, or $\widehat{r} = (\epsilon r_1, r_2)$. Factors $\delta, \epsilon \neq 1$ entail distinct totals and preclude convergence.

Example 1 Input and output are succinctly displayed through

$$A = \begin{array}{c} \\ 2 \\ 4 \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} 3 & 3 \end{array} \\ \begin{pmatrix} 30 & 0 \\ 10 & 20 \end{pmatrix} \\ \begin{array}{cc} 15 & 10 \end{array} \end{array} \begin{array}{c} \\ 1 \\ \frac{2}{3} \\ \\ \end{array} \qquad \mapsto \qquad B = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}.$$

The weight matrix A is bordered at the left by the target row marginals $r = (2,4)$, at the top by the target column marginals s, at the right by standardized row divisors, and at the bottom by column divisors. The divisors yield the biproportional fit B.

For $t = 2, 4, \ldots$ we get divisors

$$\rho_1(t-1) = \frac{6 \cdot 2^{t/2} + 6}{6 \cdot 2^{t/2} + 4}, \quad \rho_2(t-1) = \frac{6 \cdot 2^{t/2} + 3}{6 \cdot 2^{t/2} + 4},$$

and $\sigma_1(t) = 1/\rho_1(t+1)$ and $\sigma_2(t) = 1/\rho_2(t-1)$, with weight matrices

$$A(t-1) = \begin{pmatrix} 2 & 0 \\ 1 - \dfrac{1}{2 \cdot 2^{t/2} + 1} & 3 + \dfrac{1}{2 \cdot 2^{t/2} + 1} \end{pmatrix}, \quad A(t) = \begin{pmatrix} 2 + \dfrac{1}{3 \cdot 2^{t/2} + 1} & 0 \\ 1 - \dfrac{1}{3 \cdot 2^{t/2} + 1} & 3 \end{pmatrix}.$$

The IPF sequence converges to the limit B shown. The L_1-error $f\big(A(t-1)\big) = 2\big/\big(2 \cdot 2^{t/2} + 1\big) > f\big(A(t)\big) = 2\big/\big(3 \cdot 2^{t/2} + 1\big)$ tends to zero exponentially fast.

With biproportional fit B available, we standardize divisors as in the proof of Theorem 2. The first row divisor is set to unity, $u_1 = 1$. Thereafter we obtain $v_1 = 15$, and $u_2 = 2/3$, and $v_2 = 10$. The fit B is seen to be direct.

Fitting the weight matrix A to the target marginals r and s provides sufficient insight for fitting A to row marginals $\widetilde{r} = \delta r$ for some $\delta > 0$. The incremental row and column divisors and the IPF sequence take the form

$$\widetilde{\rho}_i(t-1) = \frac{1}{\delta}\rho_i(t-1), \quad \widetilde{\sigma}_j(t) = \delta\sigma_j(t),$$
$$\widetilde{A}(t-1) = \delta A(t-1), \quad \widetilde{A}(t) = A(t).$$

With $\delta \neq 1$ marginal totals no longer coincide, $\widetilde{r}_+ = 6\delta \neq 6 = s_+$, and the IPF sequence cannot converge. Two accumulation points emerge, along the subsequence of even steps $\lim_{t=0,2,\ldots} \widetilde{A}(t) = B$, and along the subsequence of odd steps $\lim_{t=1,3,\ldots} \widetilde{A}(t) = \delta B$.

The value of the L_1-error depends on the cases

$$\widetilde{\rho}_1(t-1) \leq 1, \text{ that is, } \delta \geq \rho_1(t-1), \qquad \widetilde{\rho}_2(t-1) \geq 1, \text{ that is, } \delta \leq \rho_2(t-1),$$
$$\widetilde{\sigma}_2(t) \leq 1, \text{ that is, } \delta \leq \frac{1}{\sigma_2(t)}, \qquad \widetilde{\sigma}_1(t) \geq 1, \text{ that is, } \delta \geq \frac{1}{\sigma_1(t)}.$$

The intervals $[\rho_2(t-1), \rho_1(t-1)] \supset [1/\sigma_2(t), 1/\sigma_1(t)]$ shrink to unity. Whenever they contain δ, which may happen in the beginning, the L_1-error is

$$f\big(\widetilde{A}(t-1)\big) = \frac{2\delta}{2 \cdot 2^{t/2} + 1}, \qquad f\big(\widetilde{A}(t)\big) = 2\,(\delta - 1) + \frac{2}{3 \cdot 2^{t/2} + 1}.$$

When eventually the factor $\delta \neq 1$ comes to lie outside the intervals, the L_1-error function stays put at value $|\,\widetilde{r}_+ - s_+| = 6\,|\delta - 1|$. □

Example 2 The input keeps A and r, but uses uniform row marginals:

$$\begin{array}{cccc} & 3 & 3 & \\ A = \begin{array}{c} 3 \\ 3 \end{array} & \multicolumn{2}{c}{\begin{pmatrix} 30 & 0 \\ 10 & 20 \end{pmatrix}} & \begin{array}{c} 1 \\ t \end{array} \\ & 10 & \frac{20}{3t} & \end{array} \qquad \mapsto \qquad B = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}.$$

For $t = 2, 4, \ldots$ we get divisors $\rho_1(t-1) = (t+1)/(t+2)$ and $\rho_2(t-1) = (t+3)/(t+2)$, and $\sigma_1(t) = 1/\rho_1(t+1)$ and $\sigma_2(t) = 1/\rho_2(t-1)$, with weight matrices

$$A(t-1) = \begin{pmatrix} 3 & 0 \\ \dfrac{3}{t+3} & 3 - \dfrac{3}{t+3} \end{pmatrix}, \quad A(t) = \begin{pmatrix} 3 - \dfrac{3}{t+4} & 0 \\ \dfrac{3}{t+4} & 3 \end{pmatrix}.$$

The IPF sequence tends to B. The L_1-error tends to zero linearly, $f\big(A(t)\big) = 6/(t+4)$.

Since the bottom left cell is fading, B features more zeros than A. Hence the fit is not direct, and we cannot shortcut its limit character. Setting $u_1 = 1$, the identity $30/(u_1 v_1) = 3$ yields $v_1 = 10$. Thereafter we get $u_2(t) = t$ say, and $v_2(t) = 20/(3t)$.

The fading cell decomposes the limit B into two connected components, and suggests to contemplate row marginals, $\widehat{r} = (3\epsilon, 3)$ with $\epsilon \in (0, \infty)$. Fitting A to the scaled row marginals $\widehat{r}$ and to the column marginals s, we get for $t = 2, 4, \ldots$:

$$\widehat{\rho}_1(t-1) = \frac{2\left(\sum_{n=0}^{t/2-2} \epsilon^n\right) + 3\epsilon^{t/2-2}}{2\left(\sum_{n=0}^{t/2-1} \epsilon^n\right) + 3\epsilon^{t/2-1} - 1}, \qquad \widehat{\rho}_2(t-1) = \frac{2\left(\sum_{n=0}^{t/2-1} \epsilon^n\right) + 3\epsilon^{t/2-1}}{2\left(\sum_{n=0}^{t/2-1} \epsilon^n\right) + 3\epsilon^{t/2-1} - 1},$$

and $\widehat{\sigma}_1(t) = 1/\widehat{\rho}_1(t+1)$ and $\widehat{\sigma}_2(t) = 1/\widehat{\rho}_2(t-1)$, with weight matrices

$$\widehat{A}(t-1) = \begin{pmatrix} 3\epsilon & 0 \\ \dfrac{3}{2\left(\sum_{n=0}^{t/2-1} \epsilon^n\right) + 3\epsilon^{t/2-1}} & 3 - \dfrac{3}{2\left(\sum_{n=0}^{t/2-1} \epsilon^n\right) + 3\epsilon^{t/2-1}} \end{pmatrix},$$

$$\widehat{A}(t) = \begin{pmatrix} 3 - \dfrac{3}{2\left(\sum_{n=0}^{t/2} \epsilon^n\right) + 3\epsilon^{t/2} - 1} & 0 \\ \dfrac{3}{2\left(\sum_{n=0}^{t/2} \epsilon^n\right) + 3\epsilon^{t/2} - 1} & 3 \end{pmatrix}.$$

With $\epsilon \neq 1$ two accumulation points emerge, along the even and odd subsequences. For $\epsilon > 1$ the accumulation points have the same connected components as has B,

$$\lim_{t=0,2,\ldots} \widehat{A}(t) = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \lim_{t=1,3,\ldots} \widehat{A}(t) = \begin{pmatrix} 3\epsilon & 0 \\ 0 & 3 \end{pmatrix}.$$

For $\epsilon < 1$ the geometric series gives

$$\lim_{t=0,2,\ldots} \widehat{A}(t) = \begin{pmatrix} 3 - 3\dfrac{1-\epsilon}{1+\epsilon} & 0 \\ 3\dfrac{1-\epsilon}{1+\epsilon} & 3 \end{pmatrix}, \qquad \lim_{t=1,3,\ldots} \widehat{A}(t) = \begin{pmatrix} 3\epsilon & 0 \\ \frac{3}{2}(1-\epsilon) & 3 - \frac{3}{2}(1-\epsilon) \end{pmatrix}.$$

A combination of present partial scaling with previous proportional scaling permits to consider general marginals $\widetilde{\widehat{r}} = \delta(3\epsilon, 3)$. For instance, the choices $\delta = 4/3$ and $\epsilon = 1/2$ retrieve the row marginals $(2, 4)$ and the accompanying IPF sequence of Example 1. □

Example 3 Our third example uses target row marginals $r = (4, 2)$:

$$A = \begin{array}{cc} & \begin{array}{cc} 3 & 3 \end{array} \\ \begin{array}{c} 4 \\ 2 \end{array} & \begin{pmatrix} 30 & 0 \\ 10 & 20 \end{pmatrix} \end{array}.$$

The current row marginals $(4, 2)$ may be obtained from proportional and partial scalings of the row marginals $(3, 3)$ of Example 2 by means of $\delta = 2/3$ and $\epsilon = 2$.

The divisors turn out to be, for $t = 2, 4, \ldots$:

$$\rho_1(t-1) = \frac{21 \cdot 2^{t/2} - 24}{28 \cdot 2^{t/2} - 24}, \quad \rho_2(t-1) = \frac{42 \cdot 2^{t/2} - 24}{28 \cdot 2^{t/2} - 24},$$

and $\sigma_1(t) = 1/\rho_1(t+1)$ and $\sigma_2(t) = 1/\rho_2(t-1)$. The scaled weight matrices become

$$A(t-1) = \begin{pmatrix} 4 & 0 \\ \dfrac{4}{7 \cdot 2^{t/2} - 4} & 2 - \dfrac{4}{7 \cdot 2^{t/2} - 4} \end{pmatrix}, \quad A(t) = \begin{pmatrix} 3 - \dfrac{6}{14 \cdot 2^{t/2} - 6} & 0 \\ \dfrac{6}{14 \cdot 2^{t/2} - 6} & 3 \end{pmatrix}.$$

The L_1-error is $f\big(A(t-1)\big) = 2 + \big(8/(7 \cdot 2^{t/2} - 4)\big) \geq f\big(A(t)\big) = 2 + \big(6/(7 \cdot 2^{t/2} - 3)\big)$, and tends to the limit 2 exponentially fast. The IPF procedure eventually oscillates between the accumulation points along the even and odd subsequences,

$$\lim_{t=0,2,\ldots} A(t) = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \lim_{t=1,3,\ldots} A(t) = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}. \qquad \square$$

5 The L_1-error limit

Judging from empirical evidence and experimental simulations the even-step subsequence of an IPF sequence would appear to always be convergent. Lemma 2 assembles some consequences of this hypothesis, and prepares for the L_1-error limit in Theorem 4.

Lemma 2 (Even-step subsequence limit) *For the fitting of the weight matrix A to the target marginals r and s, let $B = \lim_{n\to\infty} A(t_n)$ be an accumulation point along even steps t_n. For the fitting of B to the target marginals r and s, let $\alpha_i(1) = b_{i+}/r_i$ designate the step-one incremental row divisors, $b_{ij}(1) = b_{ij}/\alpha_i(1)$ the step-one scaling of B, and $\beta_j(2) = b_{+j}(1)/s_j$ the step-two incremental column divisors.*

Then we have:

If the even-step IPF subsequence $A(0), A(2), \ldots$ for the fitting of A to r and s converges to B, then

$$b_{ij} > 0 \implies \alpha_i(1)\beta_j(2) = 1 \qquad \text{for all } i \leq k \text{ and } j \leq \ell. \qquad [*]$$

If [*] *holds true, then* (i) *the even-step IPF subsequence $A^B(0), A^B(2), \ldots$ for the fitting of the trimmed weight matrix A^B to r and s converges to B,* (ii) *the odd-step IPF subsequence $A^B(1), A^B(3), \ldots$ converges to $B(1)$, and* (iii) *the IPF sequence for the fitting of B to r and s oscillates between $B = B(2) = \cdots$, and $B(1) = B(3) = \cdots$.*

Proof Any even-step accumulation point B qualifies as a weight matrix to be fitted to the target marginals r and s. Indeed, B has fitted columns, $b_{+j} = s_j > 0$, and so columns do not vanish. Setting $s_{\min} = \min\{s_1, \ldots, s_\ell\}$, the column divisors of A are bounded from above, $\sigma_j(t) = a_{+j}(t-1)/s_j \le a_{++}(t-1)/s_{\min} = r_+/s_{\min}$. This bounds row sums from below, $a_{i+}(t) = \sum_{j\le\ell} a_{ij}(t-1)/\sigma_j(t) \ge s_{\min} a_{i+}(t-1)/r_+ = s_{\min} r_i/r_+ > 0$, and so rows do not vanish either. With initialization $B(0) = B$, we designate the induced IPF sequence by $B(z)$, $z \ge 1$.

Assume that the even-step subsequence $A(0), A(2), \ldots$ is convergent to B. The L_1-distance between two successive even-step scaled matrices $A(t-2)$ and $A(t)$ is

$$\sum_{i\le k}\sum_{j\le\ell}\Big|a_{ij}(t-2) - a_{ij}(t)\Big| = \sum_{i\le k}\sum_{j\le\ell}\Big|\rho_i(t-1)\sigma_j(t) - 1\Big|a_{ij}(t).$$

The convergence assumption lets the L_1-distance tend to zero. Using definitions [IPF1] – [IPF4], the divisors on the right hand side are seen to converge to $\alpha_i(1)$ and $\beta_j(2)$,

$$\lim_{t=2,4,\ldots} \rho_i(t-1) = \lim_{t=2,4,\ldots} \frac{a_{i+}(t-2)}{r_i} = \frac{b_{i+}}{r_i} = \alpha_i(1),$$

$$\lim_{t=0,2,\ldots} \sigma_j(t) = \lim_{t=0,2,\ldots} \frac{a_{+j}(t-2)}{\rho_i(t-1)s_j} = \frac{b_{+j}(1)}{s_j} = \beta_j(2).$$

This yields $\sum_{i\le k}\sum_{j\le\ell}\big|\alpha_i(1)\beta_j(2) - 1\big|b_{ij} = 0$, and proves [$*$].

Conversely, assume that [$*$] holds true. We decompose B into its connected components, say

$$B = \begin{array}{c} \\ I_1 \\ I_2 \\ \vdots \\ I_K \end{array} \begin{array}{c} \begin{array}{cccc} J_1 & J_2 & \cdots & J_K \end{array} \\ \begin{pmatrix} B^{(1)} & 0 & \cdots & 0 \\ 0 & B^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B^{(K)} \end{pmatrix} \end{array}.$$

(i) Because of [$*$] each connected component $B^{(m)}$ has its row divisors equal to some common value δ. From $\sum_{i\in I_m} \delta r_i = \sum_{i\in I_m} b_{i+}$ we get $\delta = s_{J_m}/r_{I_m}$. This yields $\alpha_i(1) = s_{J_m}/r_{I_m}$ and $b_{i+} = (s_{J_m}/r_{I_m})r_i$, for all rows $i \in I_m$ and $m \le K$.

The fitting of the trimmed weight matrix A^B, with $I_m \times J_m$ blocks $A^{(m)}$ copied from A and zeros elsewhere, to r and s generates the IPF sequence $A^B(t)$, $t \ge 1$. The subsequent argument imitates the proof of Theorem 3. Let $g\left(A^B(t)\right)$ be the geometric matrix-mean of the entries $a^B_{ij}(t)$, with exponents b_{ij}/b_{++},

$$g\big(A^B(t)\big) = \prod_{i\le k}\prod_{j\le\ell} a^B_{ij}(t)^{\frac{b_{ij}}{b_{++}}} = \prod_{m\le K}\left(\prod_{i\in I_m}\prod_{j\in J_m} a^{(m)}_{ij}(t)^{\frac{b_{ij}}{s_{J_m}}}\right)^{\frac{s_{J_m}}{s_+}}.$$

Let $\rho^{(m)}_i(t-1)$ and $\sigma^{(m)}(t)$ denote the divisors for the fitting of $A^{(m)}$ to the partial marginals $(r_i)_{i\in I_m}$ and $(s_j)_{j\in J_m}$, for $m \le K$. On the way-up from an even step $t-2$ to the next even step t, we use $b_{i+} = (s_{J_m}/r_{I_m})r_i$ for $i \in I_m$ and $b_{+j} = s_j$ to obtain

$$\begin{aligned} g\big(A^B(t-2)\big) &= g\big(A^B(t)\big) \prod_{m\le K}\left(\left(\prod_{i\in I_m} \rho^{(m)}_i(t-1)^{\frac{r_i}{r_{I_m}}}\right)\left(\prod_{j\in J_m} \sigma^{(m)}_j(t)^{\frac{s_j}{s_{J_m}}}\right)\right)^{\frac{s_{J_m}}{s_+}} \\ &\le g\big(A^B(t)\big). \end{aligned}$$

The last estimate follows from the two inner geometric-arithmetic-mean inequalities,

$$\left(\prod_{i\in I_m}\rho_i^{(m)}(t-1)^{\frac{r_i}{r_{I_m}}}\right)\left(\prod_{j\in J_m}\sigma_j^{(m)}(t)^{\frac{s_j}{s_{J_m}}}\right)$$
$$\leq\left(\sum_{i\in I_m}\rho_i^{(m)}(t-1)\frac{r_i}{r_{I_m}}\right)\left(\sum_{j\in J_m}\sigma_j^{(m)}(t)\frac{s_j}{s_{J_m}}\right)=\frac{s_{J_m}}{r_{I_m}}\frac{r_{I_m}}{s_{J_m}}=1.$$

Again the even-step matrix-mean subsequence converges to a nonzero and finite value, and the limiting geometric-arithmetic-mean inequalities hold with equality. Thus all rows $i\in I_m$ have divisors tending to a common value, $\lim_{t=1,3,\ldots}\rho_i^{(m)}(t)=s_{J_m}/r_{I_m}$. Hence B is the biproportional fit of A^B to the row marginals $\big((s_{J_m}/r_{I_m})r_i\big)_{m\leq K}$ and to the column marginals s, and so is any other even-step accumulation point of the sequence $A^B(t)$, $t\geq 1$. All even-step accumulation points coinciding, the even-step subsequence $A^B(0), A^B(2),\ldots$ is convergent, and B is its limit.

(ii) The odd-step subsequence $A^B(1), A^B(3),\ldots$ satisfies $\lim_{t=1,3,\ldots}a_{ij}^{(m)}(t)=\lim_{t=2,4,\ldots}a_{ij}^{(m)}(t-2)/\rho_i^{(m)}(t-1)=b_{ij}/\alpha_i(1)=b_{ij}(1)$, for all $i\in I_m$, $j\in J_m$, and $m\leq K$. Hence it has limit $B(1)$.

(iii) Since each block $B^{(m)}$ has identical row divisors s_{J_m}/r_{I_m} and identical column divisors r_{I_m}/s_{J_m}, the IPF sequence $B(1), B(2), B(3), B(4),\ldots$ oscillates. □

Theorem 4 (L_1-Error limit) *Assume that the fitting of the weight matrix A to the target marginals r and s generates an IPF sequence in which the even-step subsequence is convergent.*

Then the limit of the L_1-error function is

$$\lim_{t\to\infty}f\big(A(t)\big)=\max_{I\subseteq\{1,\ldots,k\}}\Big(r_I-s_{J_A(I)}+s_{J_A(I)'}-r_{I'}\Big).$$

Proof Let B designate the limit of the even-step subsequence of the IPF sequence, $\lim_{t=0,2,\ldots}A(t)=B$. Denoting by $U=\{\,i\leq k\,|\,b_{i+}<r_i\,\}$ the set of rows underweighted in B, we show that $\lim_{t\to\infty}f\big(A(t)\big)=f(B)=r_U-s_{J_A(U)}+s_{J_A(U)'}-r_{U'}$.

For the fitting of B to the marginals r and s, the incremental divisors are $\alpha_i(1)=b_{i+}/r_i$ and $\beta_j(2)=b_{+j}(1)/s_j$. Underweightedness lets the rows $i\in U$ satisfy $\alpha_i(1)<1$. Complementary rows $i\in U'$ fulfill $\alpha_i(1)\geq 1$. By Lemma 2, $b_{ij}>0$ entails $\alpha_i(1)\beta_j(2)=1$, whence the columns $j\in J_B(U)$ connected in B to U have $\beta_j(2)>1$. Columns $j\in J_B(U)'\subseteq J_B(U')$ have $\beta_j(2)\leq 1$. The state of affairs may be depicted as follows:

$$B=\begin{array}{cccc} & J_B(U) & J_B(U)' & \\ \begin{matrix}U\\U'\end{matrix} & \multicolumn{2}{c}{\begin{pmatrix} B^{(1)} & \quad 0 \\ 0 & \quad B^{(2)}\end{pmatrix}} & \begin{matrix}\alpha_i(1)<1\\ \alpha_i(1)\geq 1\end{matrix} \\ & \beta_j(2)>1 & \beta_j(2)\leq 1 & \end{array}$$

The bottom left block $U'\times J_B(U)$ has $b_{ij}=0$, as otherwise $b_{ij}>0$ would imply the contradiction $1<\alpha_i(1)\beta_j(2)=1$. The top right block is zero by definition of $J_B(U)'$.

The identities $\sum_{i\in U} b_{i+} = s_{J_B(U)}$ and $\sum_{i\in U'} b_{i+} = s_{J_B(U)'}$ turn the L_1-error of B, which originates from rows only, into

$$f(B) = \sum_{i\in U}\Big(r_i - b_{i+}\Big) + \sum_{i\in U'}\Big(b_{i+} - r_i\Big) = r_U - s_{J_B(U)} + s_{J_B(U)'} - r_{U'}.$$

It remains to establish that $J_B(U) = J_A(U)$.

In the top right block $U \times J_B(U)'$ we have $\alpha_i(1)\beta_j(2) < 1$. Convergence of the even-step subsequence implies that the products $\rho_i(t-1)\sigma_j(t)$ converge to $\alpha_i(1)\beta_j(2)$, whence the cumulative divisors tend to zero, $\lim_{t\to\infty} x_i(t)y_j(t) = 0$. Now $\lim_{t\to\infty} a_{ij}/\big(x_i(t)y_j(t)\big) = b_{ij} = 0$ necessitates $a_{ij} = 0$. But $a_{ij} = 0$ on $U \times J_B(U)'$ implies $J_A(U) \subseteq J_B(U)$. The converse inclusion $J_B(U) \subseteq J_A(U)$ holds true since $b_{ij} > 0$ implies $a_{ij} > 0$. This establishes $J_A(U) = J_B(U)$. □

We are unable to prove in full generality that every IPF sequence is such that its even-step subsequence is convergent. Theorem 5 presents a partial result in this direction: The even-step IPF subsequence is convergent provided it features an accumulation point that is connected. To this end Lemma 3 recalls an intriguing result of Bacharach (1970, page 50), a succession of interlacing inequalities between incremental row and column divisors showing that the smallest of the incremental divisors gets larger and the largest gets smaller. The smallest incremental row and column divisors are denoted by $\rho_{\min}(t-1)$ and $\sigma_{\min}(t)$, and the largest by $\rho_{\max}(t-1)$ and $\sigma_{\max}(t)$.

Lemma 3 (Bacharach inequalities) *Let step $t \geq 2$ be even.*

(i) *The following interlacing inequalities hold true:*

$$\rho_{\min}(t-1) \overset{(1)}{\leq} \frac{1}{\sigma_{\max}(t)} \overset{(2)}{\leq} \rho_{\min}(t+1) \leq \frac{s_+}{r_+} \leq \rho_{\max}(t+1) \overset{(3)}{\leq} \frac{1}{\sigma_{\min}(t)} \overset{(4)}{\leq} \rho_{\max}(t-1).$$

(ii) *If A is connected and the smallest row divisors stay constant over $k-1$ subsequent row adjustments, $\rho_{\min}(t-1) = \rho_{\min}\big(t-1+2(k-1)\big)$, then all row divisors are identical, $\rho_1(t-1) = \cdots = \rho_k(t-1)$.*

Proof (i) For all rows $i \leq k$ and all columns $j \leq \ell$ we have $a_{i+}(t-1) = a_{i+}(t+1) = r_i$ and $a_{+j}(t-2) = a_{+j}(t) = s_j$. This yields

$$1 = \frac{a_{+j}(t)}{s_j} = \frac{1}{s_j}\sum_{p\leq k}\frac{a_{pj}(t-2)}{\rho_p(t-1)\sigma_j(t)} \begin{cases} \leq \dfrac{1}{\rho_{\min}(t-1)\sigma_j(t)}, & (1j) \\ \geq \dfrac{1}{\rho_{\max}(t-1)\sigma_j(t)}; & (4j) \end{cases}$$

$$1 = \frac{a_{i+}(t+1)}{r_i} = \frac{1}{r_i}\sum_{q\leq \ell}\frac{a_{iq}(t-1)}{\sigma_q(t)\rho_i(t+1)} \begin{cases} \leq \dfrac{1}{\sigma_{\min}(t)\rho_i(t+1)}, & (3i) \\ \geq \dfrac{1}{\sigma_{\max}(t)\rho_i(t+1)}. & (2i) \end{cases}$$

Forming maxima and minima over $i \leq k$ or $j \leq \ell$ in $(1j)$, $(4j)$, $(3i)$, and $(2i)$, we get

$$\rho_{\min}(t-1)\sigma_{\max}(t) \overset{(1)}{\leq} 1 \overset{(4)}{\leq} \rho_{\max}(t-1)\sigma_{\min}(t),$$

$$\sigma_{\min}(t)\rho_{\max}(t+1) \overset{(3)}{\leq} 1 \overset{(2)}{\leq} \sigma_{\max}(t)\rho_{\min}(t+1).$$

The unnumbered inequalities in the middle of the assertion follow from $\rho_{\min}(t+1) \leq \sum_{i\leq k}\rho_i(t+1)r_i/r_+ = s_+/r_+ \leq \rho_{\max}(t+1)$.

(ii) The proof is by contraposition, showing that the two assumptions $\rho_{\min}(t-1) < \rho_{\max}(t-1)$ and $\rho_{\min}(t-1) = \rho_{\min}(t+2k-3)$ force A to be disconnected. Disconnected components of A are constructed by means of the row subsets $I(z)$ where the row divisor is minimum, and the column subsets $J(z)$ where the column divisor is maximum,

$$\begin{aligned} I(z) &= \{\, i \le k \mid \rho_i(z) = \rho_{\min}(z) \,\} \qquad \text{for } z \text{ odd},\\ J(z) &= \{\, j \le \ell \mid \sigma_j(z) = \sigma_{\max}(z) \,\} \qquad \text{for } z \text{ even}. \end{aligned}$$

Due to the first assumption, $\rho_{\min}(t-1) < \rho_{\max}(t-1)$, the subset $I(t-1)$ is nonempty and proper, $\emptyset \ne I(t-1) \ne \{1,\dots,k\}$.

The second assumption expands into an equality string, $\rho_{\min}(t-1) = 1/\sigma_{\max}(t) = \rho_{\min}(t+1) = \cdots = \rho_{\min}(t+2k-5) = 1/\sigma_{\max}(t+2k-4) = \rho_{\min}(t+2k-3)$. We work our way in sets of three,

$$\rho_{\min}(t-1+z) = \frac{1}{\sigma_{\max}(t+z)} = \rho_{\min}(t+1+z), \qquad \text{with } z = 0, 2, \dots, 2k-4.$$

For $z = 0$ and $i \in I(t+1)$ we get $1/\sigma_{\max}(t) = \rho_{\min}(t+1)$. Therefore equality holds in $(2i)$, and all $q \notin J(t)$ have $a_{iq}(t) = 0$ and hence $a_{iq} = 0$. For $j \in J(t)$ equality obtains in $(1j)$, whence all $p \notin I(t-1)$ fulfill $a_{pj}(t-1) = 0$ and hence $a_{pj} = 0$. Any row $i \in I(t+1) \setminus I(t-1)$ would vanish, having $a_{ij} = 0$ for $j \in J(t)$ as well as for $j \notin J(t)$. Since vanishing rows in A are not allowed, we get $I(t+1) \subseteq I(t-1)$.

The argument carries forward to build a chain of $k-1$ inclusions,

$$\emptyset \ne I(t+2k-3) \subseteq I(t+2k-5) \subseteq \cdots \subseteq I(t+1) \subseteq I(t-1) \ne \{1,\dots,k\}.$$

At most $k-2$ inclusions can be strict. Somewhere between $z = 0$ and $z = 2k-4$ there is an equality, $I(t+1+z) = I(t-1+z)$. This forces A to be disconnected,

$$A = \begin{array}{r} \\ I(t-1+z) \\ I(t-1+z)' \end{array} \begin{array}{c} \begin{array}{cc} J(t+z) & J(t+z)' \end{array} \\ \begin{pmatrix} A^{(1)} & 0 \\ 0 & A^{(2)} \end{pmatrix} \end{array}. \qquad \square$$

Theorem 5 (Connected even-step accumulation point) *Assume that the fitting of the weight matrix A to the target marginals r and s generates an IPF sequence $A(t)$, $t \ge 1$, in which the even-step subsequence admits a connected accumulation point.*

Then the even-step IPF subsequence $A(0), A(2), \dots$ is convergent.

Proof Due to the proof of Lemma 2 the accumulation point B is a legitimate input matrix for the IPF procedure, to be fitted to the target marginals r and s. We denote the ensuing row and column divisors by $\alpha_i(z-1)$ and $\beta_j(z)$, $z = 2, 4, \dots$.

Assuming B to arise as the limit along the even-step subsequence t_n, $n \ge 1$, let the arrow $\to$ indicate a passage to the limit as n tends to infinity. Using $B(0) = B$, definitions [IPF1]–[IPF4] yield, for steps $z = 2, 4, \dots$,

$$\rho_i(t_n + z - 1) = \frac{a_{i+}(t_n + z - 2)}{r_i} \to \frac{b_{i+}(z-2)}{r_i} = \alpha_i(z-1),$$

$$a_{ij}(t_n + z - 1) = \frac{a_{ij}(t_n + z - 2)}{\rho_i(t_n + z - 1)} \to \frac{b_{ij}(z-2)}{\alpha_i(z-1)} = b_{ij}(z-1),$$

$$\sigma_j(t_n+z) = \frac{a_{+j}(t_n+z-1)}{s_j} \to \frac{b_{+j}(z-1)}{s_j} = \beta_j(z),$$

$$a_{ij}(t_n+z) = \frac{a_{ij}(t_n+z-1)}{\sigma_j(t_n+z)} \to \frac{b_{ij}(z-1)}{\beta_j(z)} = b_{ij}(z).$$

For the fitting of A to r and s, Lemma 3(i) states that the sequence of smallest row divisors, $\rho_{\min}(t)$, is monotone and bounded, and hence convergent. For the fitting of B to r and s this implies $\alpha_{\min}(z) = \lim_{n\to\infty} \rho_{\min}(t_n+z) = \lim_{t=1,3,\ldots} \rho_{\min}(t)$, for all steps $z = 1, 3, \ldots$, whence the sequence remains constant for $z \geq 1$. Lemma 3(ii) applies to the fitting of the connected matrix B to r and s, and yields $\alpha_{\min}(1) = \alpha_{\max}(1)$. Hence all rows $i \leq k$ have the same divisors $\alpha_i(1) = s_+/r_+$, and all columns the same divisors $\beta_j(2) = r_+/s_+$. Since their product is unity, Lemma 2 shows that the trimmed weight matrix A^B has its even-step subsequence converging to B. Due to connectedness of B the trimming has no effect, $A^B = A$, thus completing the proof. □

The fitting of an $I_m \times J_m$ block $A^{(m)}$ to the partial target marginals $(r_i)_{i\in I_m}$ and $(s_j)_{j\in J_m}$ yields scaled matrices $\left(A^{(m)}\right)(t)$. They are generally distinct from the blocks $\left(A(t)\right)^{(m)}$ extracted from the scaled matrices $A(t)$ for the fitting of the $k \times \ell$ matrix A to r and s. Bacharach (1970, page 53) uses the same notation for both, A^{2t}_{kk}, whence we feel that his "proof by notation" is inconclusive. The examples in Section 4 illustrate that the interrelation between $\left(A^{(m)}\right)(t)$ and $\left(A(t)\right)^{(m)}$ looks challenging.

References

Bacharach, M. (1965). Estimating nonnegative matrices from marginal data. *International Economic Review (Osaka), 6,* 294–310.

Bacharach, M. (1970). *Biproportional Matrices & Input-Output Change.* Cambridge University Press, Cambridge UK.

Balinski, M.L., & Demange, G. (1989). Algorithms for proportional matrices in reals and integers. *Mathematical Programming, 45,* 193–210.

Balinski, M.L., & Pukelsheim, F. (2006). Matrices and politics. In *Festschrift for Tarmo Pukkila on His 60th Birthday,* (Eds. E. Liski, J. Isotalo, S. Puntanen, G.P.H. Styan), Department of Mathematics, Statistics, and Philosophy, University of Tampere, 233–242

Balinski, M.L., & Rachev, S.T. (1997). Rounding proportions: Methods of rounding. *Mathematical Scientist, 22,* 1–26.

Berge, C. (1985). *Graphs. Second Revised Edition.* North-Holland, Amsterdam.

Bishop, Y.M.M., Fienberg, S.E., & Holland, P.W. (1975). *Discrete Multivariate Analysis: Theory and Practice.* MIT Press, Cambridge MA.

Bregman, L.M. (1967). Proof of the convergence of Sheleikhovskii's method for a problem with transportation constraints. *USSR Computational Mathematics and Mathematical Physics, 7,* 191–204.

Brown, D.T. (1959). A note on approximations to discrete probability distributions. *Information and Control, 2,* 386-392

Brown, J.B., Chase, P.J., & Pittinger, A.O. (1993). Order independence and factor convergence in iterative scaling. *Linear Algebra and Its Applications, 190,* 1–38.

Brualdi, R.A., Parter, S.V., & Schneider, H. (1966). The diagonal equivalence of a nonnegative matrix to a stochastic matrix. *Journal of Mathematical Analysis and Applications, 16,* 31–50.

Csiszár, I. (1975). *I*-Divergence geometry of probability distributions and minimization problems. *Annals of Probability, 3,* 146–158.

Deming, W.E., & Stephan, F.F. (1940). On a least squares adjustment of a sampled frequency table when the expected marginal totals are known. *Annals of Mathematical Statistics, 11,* 427–444.

Fagan, J.T., & Greenberg, B. (1987). Making tables additive in the presence of zeros. *American Journal of Mathematical and Management Sciences, 7,* 359-383.

Fienberg, S.E. (1970). An iterative procedure for estimation in contingency tables. *Annals of Mathematical Statistics, 41* 907–917.

Fienberg, S.E., & Meyer, M.M. (2006). Iterative proportional fitting. *Encyclopedia of Statistical Sciences, 6,* 3723–3726.

Gaffke, N., & Pukelsheim, F. (2008a). Divisor methods for proportional representation systems: An optimization approach to vector and matrix apportionment problems. *Mathematical Social Sciences, 56,* 166–184.

Gaffke, N., & Pukelsheim, F. (2008b). Vector and matrix apportionment problems and separable convex integer optimization. *Mathematical Methods of Operations Research, 67,* 133–159.

Ireland, C.T., & Kullback, S. (1968). Contingency tables with given marginals. *Biometrika, 55,* 179–188.

Joas, B. (2005). *A Graph-theoretic Solvability Check for Biproportional Multiplier Methods.* Thesis, Institut für Mathematik, Universität Augsburg.

Kalantari, B., Lari, I., Ricca, F., & Simeone, B. (2008). On the complexity of general matrix scaling and entropy minimization via the RAS algorithm. *Mathematical Programming Series A, 112,* 371–401.

Kullback, S. (1966). An information-theoretic derivation of certain limit relations for a stationary Markov chain. *SIAM Journal on Control, 4,* 454–459.

Kullback, S. (1968). Probability densities with given marginals. *Annals of Mathematical Statistics, 39,* 1236–1243.

Macgill, S.M. (1977). Theoretical properties of biproportional matrix adjustments. *Environment and Planning A, 9,* 687–701.

Maier, S. (2009). *Biproportional Apportionment Methods: Constraints, Algorithms, and Simulation.* Hut-Verlag, München.

Maier, S., Zachariassen, P., & Zachariasen, M. (2010). Divisor-based biproportional apportionment in electoral systems: A real-life benchmark study. *Management Science, 56,* 373–387.

Marshall, A.W., & Olkin, I. (1968). Scaling of matrices to achieve specified row and column sums. *Numerische Mathematik, 12,* 83–90.

Oelbermann, K.-F., & Pukelsheim, F. (2011). Future European Parliament elections: Ten steps towards uniform procedures. *Zeitschrift für Staats- und Europawissenschaften – Journal for comparative Government and European policy, 9,* 9–28.

Pennisi, A. (2006). The Italian bug: A flawed procedure for bi-proportional seat allocation. Pages 151–166 in: Simeone and Pukelsheim (2006).

Pretzel, O. (1980). Convergence of the iterative scaling procedure for non-negative matrices. *Journal of the London Mathematical Society, 21,* 379–384.

Pukelsheim, F. (1998). Rounding tables on my bicycle. *Chance, 11,* 57–58.

Pukelsheim, F. (2004). BAZI—A Java program for proportional representation. *Oberwolfach Reports, 1,* 735–737.

Pukelsheim, F., & Schuhmacher, C. (2004). Das neue Zürcher Zuteilungsverfahren für Parlamentswahlen. *Aktuelle Juristische Praxis—Pratique Juridique Actuelle, 13,* 505–522.

Pukelsheim, F., & Schuhmacher, C. (2011). Doppelproporz bei Parlamentswahlen – ein Rück- und Ausblick. *Aktuelle Juristische Praxis—Pratique Juridique Actuelle, 20,* 1581–1599.

Pukelsheim, F., Ricca, F., Simeone, B., Scozzari, A., & Serafini, P. (2012). Network flow methods for electoral systems. *Networks, 59,* 73–88.

Rachev, S.T., & Rüschendorf, L. (1998). *Mass Transportation Problems.* Springer-Verlag, New York.

Ramírez, V., Pukelsheim, F., Palomares, A., & Martínez, J. (2008). The bi-proportional method applied to the Spanish Congress. *Mathematical and Computer Modelling, 48,* 1461–1467.

Rockafellar, R.T. (1984). *Network, Flows, and Monotropic Optimization.* Wiley, New York.

Rote, G., & Zachariasen, M. (2007). Matrix scaling by network flow. Pages 848–854 in: *Proceedings of the Eighteenth Annual ACM-SIAM Symposium on Discrete Algorithms, Proceedings in Applied Mathematics, 125.*

Rüschendorf, L. (1995). Convergence of the iterative proportional fitting procedure. *Annals of Statistics, 23,* 1160–1174.

Rüschendorf, L., & Thomsen, W. (1993). Note on the Schrödinger equation and I-projections. *Statistics & Probability Letters, 17,* 369–375.

Rüschendorf, L., & Thomsen, W. (1997). Closedness of sum spaces and the generalized Schrödinger problem. *Theory of Probability and Its Applications, 42,* 483–494.

Schneider, M.H. (1990). Matrix scaling, entropy minimization, and conjugate duality (II): The dual problem. *Mathematical Programming, 48,* 103–124.

Simeone, B., & Pukelsheim, F. (2006). *Mathematics and Democracy—Recent Advances in Voting Systems and Collective Choice.* Springer-Verlag, Berlin.

Sinkhorn, R. (1964). A relationship between arbitrary positive matrices and doubly stochastic matrices. *Annals of Mathematical Statistics, 35,* 877–879.

Sinkhorn, R. (1966). A relationship between arbitrary positive matrices and stochastic matrices. *Canadian Journal of Mathematics, 18,* 303–306.

Sinkhorn, R. (1967). Diagonal equivalence to matrices with prescribed row and column sums. *American Mathematical Monthly, 74,* 403–405.

Sinkhorn, R. (1972). Continuous dependence on A in the D_1AD_2 theorems. *Proceedings of the American Mathematical Society, 32,* 395–398.

Sinkhorn, R. (1974). Diagonal equivalence to matrices with prescribed row and column sums, II. *Proceedings of the American Mathematical Society, 45,* 195–198.

Sinkhorn, R., & Knopp, P. (1967). Concerning nonnegative matrices and doubly stochastic matrices. *Pacific Journal of Mathematics, 21,* 343–348.

Speed, T.P. (2005). Iterative proportional fitting. *Encyclopedia of Biostatistics, 7,* 2646–2650.

Stephan, F.F. (1942). An iterative method of adjusting sample frequency tables when expected marginal totals are known. *Annals of Mathematical Statistics, 13,* 166–178.

Wainer, H. (1998). Visual revelations: Rounding tables. *Chance, 11,* 46–50.

Zachariassen, P., & Zachariasen, M. (2006). A comparison of electoral formulae for the Farœse Parliament. Pages 235–252 in: Simeone and Pukelsheim (2006).